Reading group

Knowledge Growth and the Allocation of Time

Lucas and Moll (2014)

Journal of Political Economy (JPE)

Presented by Duong Tung Bui *(Western University)*

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Last time

Endogenous growth model

- Growth comes from (i) capital accumulation, (ii) innovation, (iii) technology adoption, and (iv) human capital and fertility
- We talked about Innovation-Based Growth, where growth stems from innovation or discovery. Examples are
    - Romer (1990): Growth in this model is driven by technological change that arises from intentional investment decisions made by profit-maximizing agents.
    - Aghion and Howitt (1992): growth comes from new innovations that replace old technologies
    - Grossman and Helpman (1991): Innovation comes directly from improvements in product quality through innovation and imitation in international trade.

Today's paper **Lucas and Moll (2014)**: growth comes from diffusion of knowledge through human interaction.

General issues and why is it important

General issues: Study the a new model of endogenous growth, driven by improvement in individual knowledge

This is important because:

- It offers a more realistic view of knowledge acquisition. Unlike traditional models where knowledge is a public good, this research treats it as "partially rival."
- There is external effect when someone gains knowledge, it also makes it easier for others around them to learn and become more productive. Because people don't consider this wider benefit, the overall social return to learning is higher than the private return. $\Rightarrow$ Implication for policy maker to design tax and subsidy to align private incentives with social benefits

Unanswered questions vs innovation

Unanswered questions:

- Existing "exogenous" or "endogenous" growth theory ignores the distinction between a person's individual effort and their learning environment, when investigating individual learning.
- Analyze the economics model where there is simultaneous determination of individual behavior and the evolution of the learning environment.

Innovation:

- Provide distinction between learning effort and learning environment in a person's leanring process
- It models knowledge as "partially rival":
  - Rival in the short run because ppl must make effort to search for knowledge and need luck to run into right ppl
  - Non-rival in the long run because will enventually be spread among all ppl

1. Introduction
2. Decentralized Economy
   - Model
   - Balance growth path
3. An Optimally Planned Economy
4. Extensions

Setup

- A mass of infinitely lived agents, each with productivity $z \sim F(z, t) = \Pr\{\tilde{z} \leq z \text{ at date } t\}$.
- Each with 1 unit of labor per t, they allocation their time to either search or working
- Individual earning:

$$y(z, t) = [1 - s(z, t)]z$$

  where $s(z, t)$ is the fraction of time searching
- Per capita GDP

$$Y(t) = \int_0^\infty [1 - s(z, t)] z f(z, t)\, dz$$

- Individual preference:

$$V(z, t) = \mathbb{E}_t \left\{ \int_t^\infty e^{-\rho(\tau - t)} [1 - s(\tilde{z}(\tau), \tau)] \tilde{z}(\tau)\, d\tau \,\middle|\, \tilde{z}(t) = z \right\}.$$

Evolution of search

Evolution of search: the process of (asymmetric) meeting btwn ppl, comparing ideas and improve their own productivity

- Over $(t, t+\Delta)$, meet one other agent with productivity z', with probability $\alpha[s(z,t)]\Delta$. where α is a function
- The agent leaves the meeting with the best productivity

$$z(t+\Delta) = \max\{z', z(t)\}$$

Evolution of search (Con't)

Evolution of search (Con't):

- We want to look at how distribution change overtime so we take $\frac{\partial F(z,t)}{\partial t}$, then differentiate with respect to z to obtain

$$\frac{\partial f(z,t)}{\partial t} = \left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{out}} + \left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{in}}$$

This representes the density of agent at productivity level z. The intuition is:

  - If productivity of agent is $y > z$, they will learn and move from z to $y \Rightarrow$ **outflow** of agent at productivity level z
  - If productivity of agent is $y < z$, they will learn move from y to $z \Rightarrow$ **inflow** of agent at productivity level z

Evolution of search (Con't)

With some math, the explicit formula for the Evolution of search is

$$\frac{\partial f(z,t)}{\partial t} = \underbrace{-\alpha(s(z,t))f(z,t)\int_z^{\infty} f(y,t)\,dy}_{\text{Outflow}}$$
$$\underbrace{+f(z,t)\int_0^{z} \alpha(s(y,t))f(y,t)\,dy}_{\text{Inflow}}$$

Where:

- $f(z,t)$: density of ppl at productivity z
- $\int_z^{\infty} f(y,t)\,dy = 1 - F(z,t)$: probability of an agent with productivity above z
  - $\int_z^{\infty} \alpha(s(z,t))f(y,t)$: probability of adopting productivity y, given agent current level is z (Outflow case)
  - $\int_0^{z} \alpha(s(y,t))f(y,t)$: probability of adopting productivity z, given agent current level is y (Inflow case)

Individual problem

Each individual has a productivity level z. At time t, their objective is to maximize the present discounted value of income. They face a tradeoff between

- **Production**: spend $1 - s(z, t)$ time working and earn $y(z, t) = [1 - s(z, t)]z$
- **Search/learning**: If they spend fraction $s(z, t)$ of their time searching

The meeting process transfers knowledge/productivity across agents, generating endogenous growth.

Individual problem (Con't)

Bellman equation

$$\rho V(z,t) - \frac{\partial V(z,t)}{\partial t} = \max_{s \in [0,1]} \Big\{ (1-s)z + \alpha(s) \int_z^{\infty} [V(y,t) - V(z,t)] f(y,t)\, dy \Big\}.$$

- Left-hand side: Standard form of HJB (Hamilton-Jacobi-Bellman) equation (More on HJB):
  - $\rho V(z,t)$: discounting
  - $-\frac{\partial V}{\partial t}$: adjustment because the environment $f(y,t)$ changes over time
- Right-hand side: The maximum payoff from choosing s:
  - $(1-s)z$: earning when allocating time $1-s$ to production
  - $\alpha(s)$: probability of meeting, increasing in s
  - $\int_z^{\infty} [V(y,t) - V(z,t)] f(y,t)\, dy$: expected gain from meeting a more productive agent.

Equiblbrium

Given the initial distribution $f(z,0)$, a (Decentralized) **Equilibrium** consist of

- Value function: $V(z,t)$; Policy function: $s(z,t)$; Distribution of agents: $f(z,t)$

such that

1. Individual solve recursive problem

$$\rho V(z,t) - \frac{\partial V(z,t)}{\partial t} = \max_{s\in[0,1]} \left\{ (1-s)z + \alpha(s)\int_z^{\infty} [V(y,t) - V(z,t)] f(y,t)\,dy \right\}.$$

2. Given s, the evaluation of search (a.k.a law of motion for distribution) satisfies

$$\frac{\partial f(z,t)}{\partial t} = -\alpha(s(z,t)) f(z,t) \int_z^{\infty} f(y,t)\,dy + f(z,t)\int_0^z \alpha(s(y,t)) f(y,t)\,dy$$

Balance growth path

Terminology

- In old models that have not incorporate technological progress, we have long-run equilibrium characterized as a constant per capita ratio (eg: Solow model). This situation is described as the **Steady state**
- When Romer and endogenous growth models came along, per capita terms were not constant in the long-run equilibrium, but growing at a constant rate
- What happens is that all per capita magnitudes grow at a balanced rate (i.e at the same rate, and so their ratios remain constant)
- Thus then "**Balance growth path**" term is used for endogenous growth model.

Balance growth path (Con't)

A balanced growth path (BGP) is a number γ and a triple of functions (ϕ, σ, v) on $\mathbb{R}_+$ such that

$$
\begin{aligned}
f(z,t) &= e^{-\gamma t}\phi(ze^{-\gamma t}),\\
V(z,t) &= e^{\gamma t}v(ze^{-\gamma t}),\\
s(z,t) &= \sigma(ze^{-\gamma t})
\end{aligned}
$$

- Intuitively: all z-quantiles grow at γ.

Simulation

Goal: Measure how sensitive the change in parameter affect the policy function

Notation

- γ: Growth rate of total production
- $\phi(\cdot), v(\cdot), \sigma(\cdot)$: time-invariant functions of the rescaled state $ze^{-\gamma t}$. They describe the stationary "shape" of the distribution, value function, and strategy.
  - The rescale trick with z multiply by economy growth trend help stabilizes the analysis. Intuitively, instead of tracking productivities that grow without bound, we work with relative productivities
- θ: Productivity distribution $F(z,0)$ has Pareto tail parameter of $\frac{1}{\theta}$. Higher θ means higher variance and flatter tail $\Rightarrow$ there are more people with high productivity to learn from

Simulation (cont.) - Pareto distribution

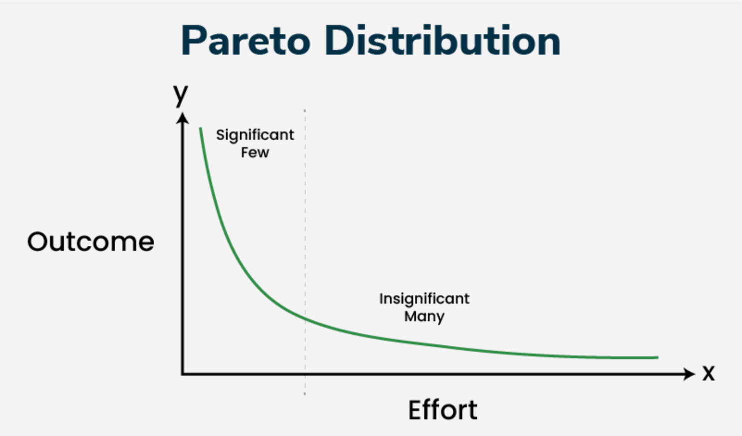

Figure 1: Source: `https://www.geeksforgeeks.org/engineering-mathematics/pareto-distribution/`

Simulation (cont.) - Pareto distribution

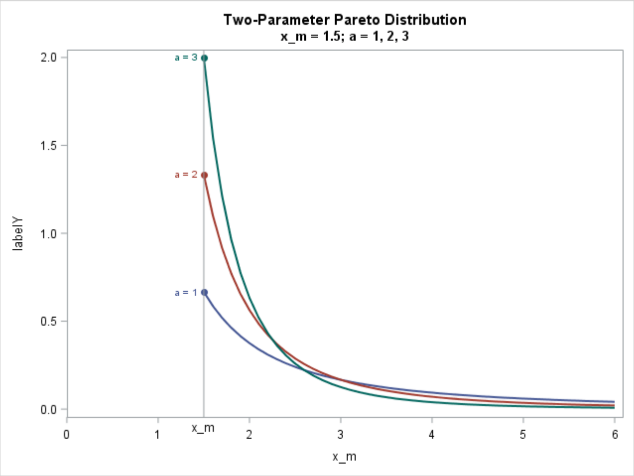

Figure 2: Source: `https://blogs.sas.com/content/iml/2018/11/05/fit-pareto-distribution-sas.html`

Simulation (cont.)

- At both case of θ, least productive agents search full-time, while most productive ones work full-time
- Low productive ppl spend more time to search bc the marginal benefit of searching is higher than producing
- Higher θ means higher return to search so ppl devote more time to search in $\theta = 0.8$ case than 0.5 at a given level of z

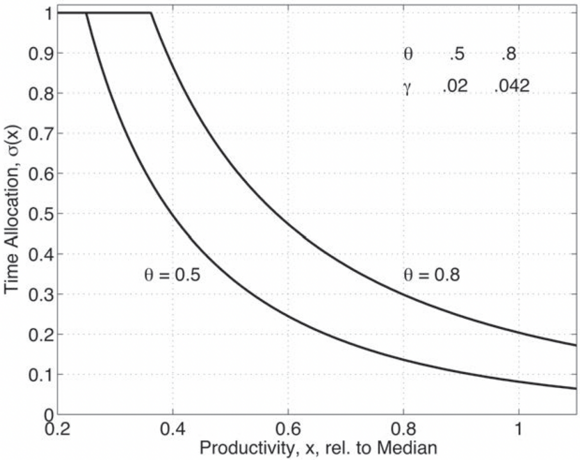

FIG. 1.—Optimal time allocation, $\sigma(x)$, for $\theta = 0.5$ and $\theta = 0.8$

Simulation (cont.)

- Lorenz Curve (LC) show the level of inequality in output/income
- Income LC represents inequality in current income, ignoring worker's ability to learn from others' productivity
- Value LC represents inequality in lifetime expected earnings, allowing low productivity workers to level up

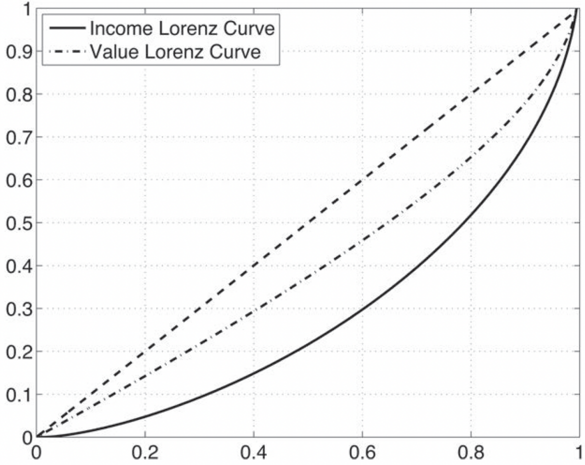

FIG. 3.—Income and value Lorenz curves for $\theta = 0.5$

Simulation (cont.)

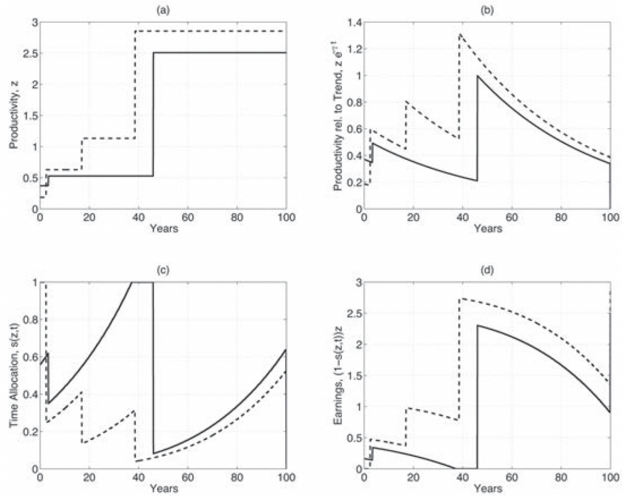

FIG. 5.—Two sample paths

An Optimally Planned Economy

- There is an externality of search in this economy and individual have not accounted for in their maximization problem
- This externality is the benefit others receive when the agent increase his knowledge
- Hence, the equilibrium defined in the Decentralized equilibrium is not efficient
- This section work out the efficient allocation through social planner problem, where there is a hypothetical, beneficent planner that allocates resources.

Social Planner Problem

- Goal: The social planner's objective is to maximize the total welfare of the economy over an infinite time horizon
- The planner decides the optimal allocation of time between producing and searching

Social Planner Problem:

$$W[f(\cdot,t)] = \max_{s(\cdot,\cdot)} \int_t^\infty e^{-\rho(\tau-t)} \int_0^\infty [1-s(z,\tau)] z f(z,\tau)\, dz d\tau$$

subject to the law of motion for f:

$$\frac{\partial f(z,\tau)}{\partial \tau} = -\alpha(s(z,\tau)) f(z,\tau) \int_z^\infty f(y,\tau) dy + f(z,\tau)\int_0^z \alpha(s(y,\tau)) f(y,\tau) dy$$

and with $f(\cdot,t)$ given.

Social Planner Problem (Con't)

Break down of terms in the objective function:

$$W[f(\cdot,t)] = \max_{s(\cdot,\cdot)} \int_t^{\infty} e^{-\rho(\tau-t)} \int_0^{\infty} [1-s(z,\tau)] z f(z,\tau)\, dz d\tau$$

Where

- $\int_t^{\infty} e^{-\rho(\tau-t)}[\ldots]d\tau$: This is an integral over an infinite time horizon.
- $\int_0^{\infty}[\ldots]dz$: This is an integral over the population of agents
- $[1-s(z,\tau)]z$: individual's earning

Social Planner Problem (Con't)

How to solve this problem, especially when the state variable is a distribution $f(\cdot, t)$.

Note

- Why state variable is a distribution?
  - State variable is known, relevant and has an evolutionary path. $f(\cdot, t)$ satisfy these properties
  - There is infinite agents, differing by productivity. Then, we care about the aggregate state, which is captured by the distribution of agents/productivity $f(\cdot, t)$
- Why does it cause a problem?
  - The state variable now is not a scalar, but multi-dimensional. This is because $f(\cdot, t)$ function that map every productivity level z to density $f(z, t)$

Social Planner Problem (Con't)

Lucas and Moll proposed a simplification as follows. Rather than solving the Bellman directly, they derive a simpler expression for the marginal social value of an individual of type z at time t which we denote by $w(z,t)$

- Define functional derivative of W,

$$\tilde{w}(z,f) = \frac{\delta W(f)}{\delta f(z)}$$

- Now define

$$w(z,t) = \tilde{w}(z, f(z,t))$$

Social Planner Problem (Con't)

The marginal social value of type z individual, $w(z,t)$ satisfy this Bellman equation:

$$\rho w(z,t) - \frac{\partial w(z,t)}{\partial t} = \max_{s\in[0,1]} \Big\{ (1-s)z \underbrace{+\alpha(s)\int_z^\infty [w(y,t)-w(z,t)]f(y,t)\,dy}_{\text{Internal benefit from search}} \underbrace{-\int_0^z \alpha(s(y,t))[w(y,t)-w(z,t)]f(y,t)\,dy \Big\}}_{\text{External benefit from search}}$$

- Problem reduce from infinite-dimensional to two dimensional
- 2nd term: expected value of improvement of type z's future productivity to $y > z$
- 3rd term: expected value of improvement in productivity of type $y < z$ to z in case they meet z

Social Planner Problem (Con't)

Back to the Bellman equation in Decentralized economy case:

$$\rho V(z,t) - \frac{\partial V(z,t)}{\partial t} = \max_{s\in[0,1]} \Big\{(1-s)z + \alpha(s)\int_z^\infty [V(y,t)-V(z,t)]f(y,t)dy\Big\}.$$

- The 3rd term (a.k.a external benefit from search) does not show up in here
- The social planner captured the 3rd term because values this external benefit, while the agent does not

SPP solution

- The planner's optimal choice of search intensity satisfies

$$\text{FOC:} \quad z = \alpha'(s(z,t)) \int_z^\infty [w(y,t) - w(z,t)] f(y,t)\, dy.$$

  The planner trades off costs and benefits from changing individual search intensities, $s(z,t)$.

- Increasing $s(z,t)$ has three effects.
  1. Production decreases by z
  2. Outflow of people at z increases by $\alpha'(s(z,t))$, corresponding to a loss

  $$-\alpha'(s(z,t))\, w(z,t) \int_z^\infty f(y,t)\, dy.$$

  3. Inflow of people into $y > z$ increases by $\alpha'(s(z,t))$. This corresponds to a gain

  $$\alpha'(s(z,t)) \int_z^\infty w(y,t) f(y,t)\, dy.$$

SPP solution (Con't)

- The planner's optimal choice of search intensity satisfies

$$\text{FOC:} \quad z = \alpha'(s(z,t)) \int_z^\infty [w(y,t) - w(z,t)] f(y,t)\, dy.$$

  Note that the RHS of FOC only integrate over $y > z$ because changing $s(z,t)$ has no direct effect on the distribution over $y < z$, which depends only on the search intensity, $s(y,t)$ of individual with productivity $y < z$

Balanced growth path

- As in decentralized problem, can restate the equations in terms of relative productivities

$$f(z,t) = e^{-\gamma t}\phi(ze^{-\gamma t}), \quad w(z,t) = e^{\gamma t}\omega(ze^{-\gamma t})$$

- We continue compare features of BGP of optimal allocation to the BGP of the decentralized equilibrium studied earlier

Simulation

- Compare allocation chosen by social planner with outcome of decentralized economy
- Planner assigns higher fraction of time to search for all agent to internalize external benefit of search
- This implies higher γ in SPP compare to decentralized economy

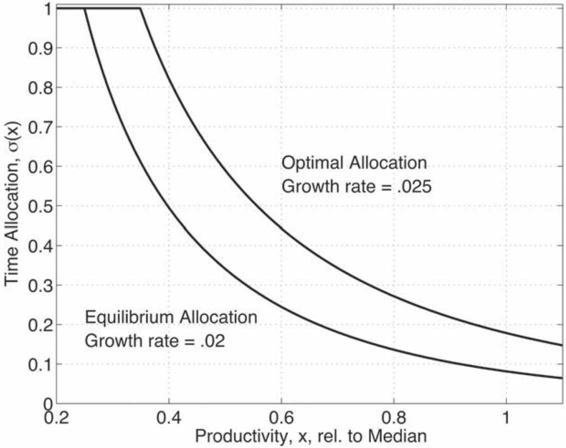

FIG. 6.—Optimal time allocation, $\sigma(x)$, in decentralized equilibrium and $\varsigma(x)$ in the planning problem.

Simulation (Con't)

- Flow income Lorenz curve: inequality in current output/income
- In planner problem, more time allocated to search implies higher income inequality - **WHY?**

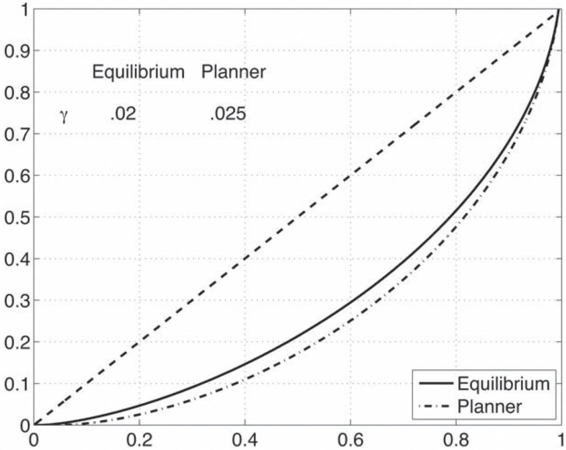

FIG. 7.—Income Lorenz curves and growth rate, γ, in decentralized equilibrium and the planning problem.

Simulation (Con't)

- Flow income Lorenz curve: inequality in current output/income
- In planner problem, more time allocated to search implies higher income inequality - **WHY?**
- Bc planner puts more time into searching to internalize externality, meaning fewer people are producing at a given moment. Thus, output is more concentrated among those who are already productive.

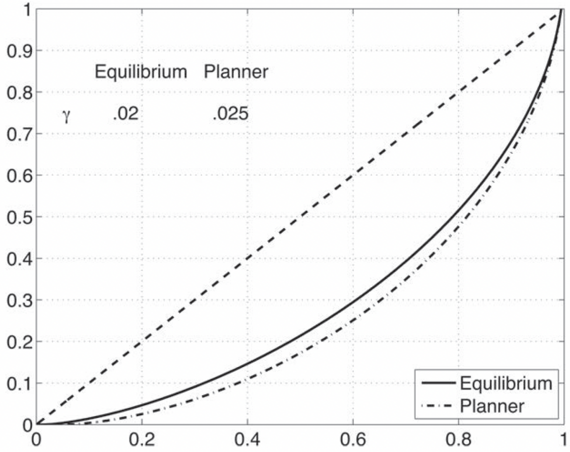

Fig. 7.—Income Lorenz curves and growth rate, γ, in decentralized equilibrium and the planning problem.

Simulation (Con't)

- Value Lorenz curve: inequality in lifetime expected earnings, which includes the effect of future mobility.
- The value Lorenz curve accounts for the fact that low-productivity people who are currently searching will move up over time.
- That future mobility reduces inequality in lifetime prospects.

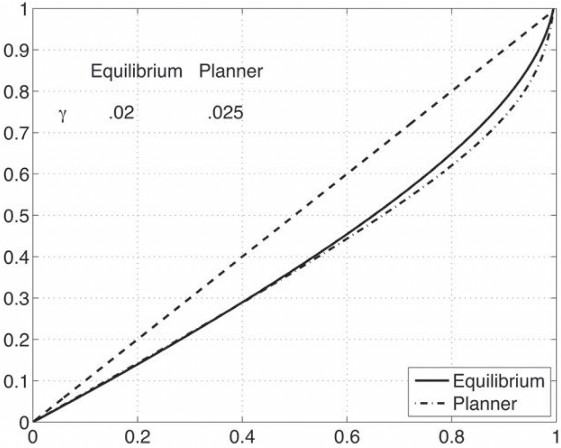

FIG. 8.—Present value Lorenz curves and growth rate, γ, in decentralized equilibrium and the planning problem.

Tax Implementation of the Optimal Allocation

- This section illustrates a tax structure to implement optimal allocation by aligning private and social return to search
- We use tax to finance productivity-related subsidy $\tau(z,t)$ to offset of opportunity cost z of search time s

Tax Implementation of the Optimal Allocation (Con't)

- The flat rate satisfies government budget

$$\underbrace{\int_0^\infty \tau(z,t)s(z,t)zf(z,t)\,dz}_{\text{Total subsidy}} = \underbrace{\tau_0\int_0^\infty [1-s(z,t)]zf(z,t)\,dz}_{\text{Total tax revenue}}$$

- Individual Bellman equation become

$$\rho V(z,t) - \frac{\partial V(z,t)}{\partial t} = \max_{s\in[0,1]}\Bigg\{\underbrace{(1-\tau_0)(1-s)z}_{\text{Income after tax}} + \underbrace{\tau(z,t)zs}_{\text{Subsidy from search/learning activities}} + \alpha(s)\int_z^\infty [V(y,t)-V(z,t)]f(y,t)\,dy\Bigg\}.$$

Tax Implementation of the Optimal Allocation (Con't)

Balance growth rate

$$f(z,t) = e^{-\gamma t}\phi(ze^{-\gamma t}),$$
$$V(z,t) = (1-\tau_0)e^{\gamma t}v_n(ze^{-\gamma t}).$$
$$s(z,t) = \sigma(ze^{-\gamma t})$$

This function $v_n(x)$ satisfies

$$(\rho-\gamma)v_n(x) + v_n'(x)\gamma x = \max_{\sigma\in[0,1]} \Big\{(1-\sigma)x + \tau(x)x\sigma + \alpha(\sigma)\int_x^\infty [v_n(y) - v_n(x)]\phi(y)\,dy\Big\}$$

Notation for simulation

- $\sigma(x)$: policy function under decentralized economy
- $\varsigma(x)$: policy function under social planner problem
- $\tau(x)$: subsidy rate

Simulation

- Interval A: agent search full time (i.e $\sigma = 1$ in both decentralized and planner case $\Rightarrow$ no tax needed to encourage more search
- Interval B: Planner wants everyone to search, so tax $\tau(x)$ is chosen to encourage search over produce
- Interval C: planner wants everyone to search, but the opportunity cost of search keeps increasing, so $\tau(x)$ needs to be an increasing func

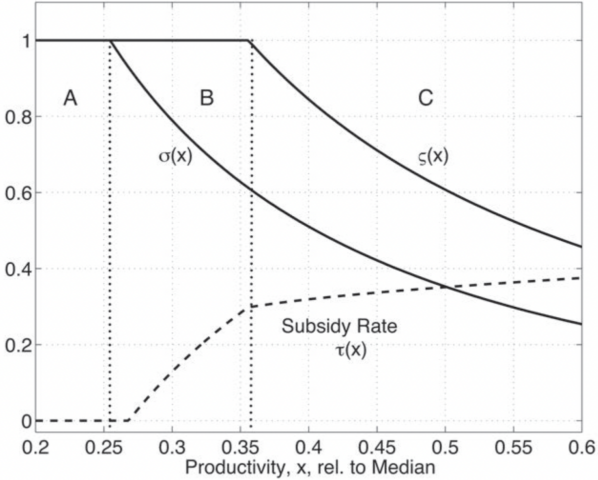

Fig. 9.—Pigovian implementation of the optimal allocation

Exogenous Knowledge Shock

- In the first version of the model, all possible knowledge was “already in the world” from the start, and ppl "copy" productivity from each other ⇒ sound unrealistic bc it ignores innovation.
- In this part, they add another version where new ideas really can emerge — but in the long run, this doesn’t change the overall shape of the productivity distribution or the growth pattern.

Exogenous Knowledge Shock (Con't)

Benchmark case

- People learn from existing idea, and the law of motion for productivity distribution is:

$$\frac{\partial F(z,t)}{\partial t} = -\alpha[1 - F(z,t)]F(z,t),$$

- At time $t = 0$, assume the productivity distribution $F(z,0)$ has a **Pareto tail** with parameter $1/\theta$. This implies that on the **balanced growth path (BGP)**:
  - Growth rate is

$$\gamma = \alpha\theta,$$

  - And the long-run productivity distribution has the form:

$$\lim_{t\to\infty} F(xe^{\gamma t}, t) = \frac{1}{1 + kx^{-1/\theta}}.$$

- This says the shape of the productivity distribution stays Pareto-like over time

Exogenous Knowledge Shock (Con't)

Extension to new idea

- Ppl not only learn from existing ideas, but also discover brand new ones.
- Let $G(z)$ = the CDF of the distribution of new ideas .
- Let β = rate at which people draw from this pool of new ideas.

The new law of motion for $F(z, t)$ is:

$$\frac{\partial F(z,t)}{\partial t} = -\alpha[1 - F(z,t)]F(z,t) - \beta[1 - G(z)]F(z,t))$$

Exogenous Knowledge Shock (Con't)

Extension to new idea (Con't): This extension offer several possiblity

1. Neither $F(z,0)$ nor $G(z)$ has Pareto tail, then

$$\lim_{t\to\infty} F(xe^{\gamma t}, t) = 1$$

meaning over time, all individual has $z < xe^{\gamma t}$

2. $F(z,0)$ has flatter tail than $G(z)$, the process converge to BGP at rate $\gamma = \alpha\theta$ and asymptotic distribution is

$$\lim_{t\to\infty} F(xe^{\gamma t}, t) = \frac{1}{1 + kx^{-1/\theta}},$$

meaning external ideas become irrelevant over time

3. $G(z)$ has flatter tail than $F(z,0)$, with $G(z)$ having tail parameter $\frac{1}{\xi}$:

$$\lim_{t\to\infty} F(xe^{\gamma t}, t) = \frac{1}{1 + (\beta/\alpha)mx^{-1/\xi}}, \quad \gamma = \alpha\xi$$

meaning external ideas matter over time

Limit to Learning

- There is a limit to the agent's ability to acquire new knowledge.
  - If y meets $z > y$ at t, he can adopt z with given probability $k(\frac{y}{z})$.
  - with prob $1 - k(\frac{y}{z})$ he cannot do this; retains his previous productivity z.

Limit to Learning (Con't)

- Law of motion is

$$\frac{\partial f(z,t)}{\partial t} = f(z,t)\int_0^z \alpha(s(y,t))f(y,t)k(z/y)\,dy$$
$$-\alpha(s(z,t))f(z,t)\int_z^\infty f(y,t)k(y/z)\,dy.$$

- Bellman equation

$$\rho V(z,t) - \frac{\partial V(z,t)}{\partial t} = \max_{s\in[0,1]} \Big\{\log[(1-s)z]$$
$$+\alpha(s)\int_z^\infty [V(y,t)-V(z,t)]k(y/z)f(y,t)\,dy\Big\}.$$

Limit to Learning (Con't)

- Balance growth path:

$$(\rho-\gamma)v(x)+v'(x)\gamma x=\max_{\sigma\in[0,1]}\Big\{\log[(1-\sigma)x]$$
$$+\alpha(\sigma)\int_x^\infty[v(y)-v(x)]k(y/x)\phi(y)dy\Big\}$$

- Growth rate of economy:

$$\gamma=\theta\delta\int_0^\infty\alpha(\sigma(y))\phi(y)dy.$$

Simulation

- In general, all productivity types allocate less time to search. This depress growth because growth depends on avg individual search intensity
- With higher limit to learning, κ, low productivity agent are discourage from search because the benefit from meeting a more productive agent is low. High productive agent also do not search bc of high opportunity cost

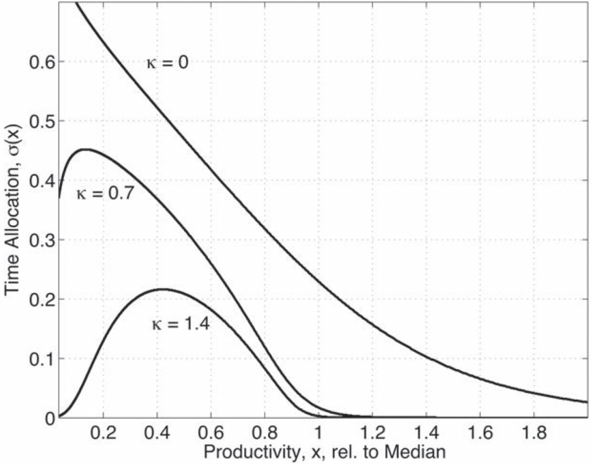

FIG. 10.—Optimal time allocation, $\sigma(x)$, for various κ values

Simulation

Discussion about how high κ affect on

- On-the-job human capital accumulation
- Social Mobility

Symmetric Meetings

View more on Lecture 9 - Benjamin Moll's Note: https://benjaminmoll.com/wp-content/uploads/2020/02/Lecture9_ECO521_web.pdf

Thank You